RBE 2004

ระบบอัตโนมัติ (Automatic System)

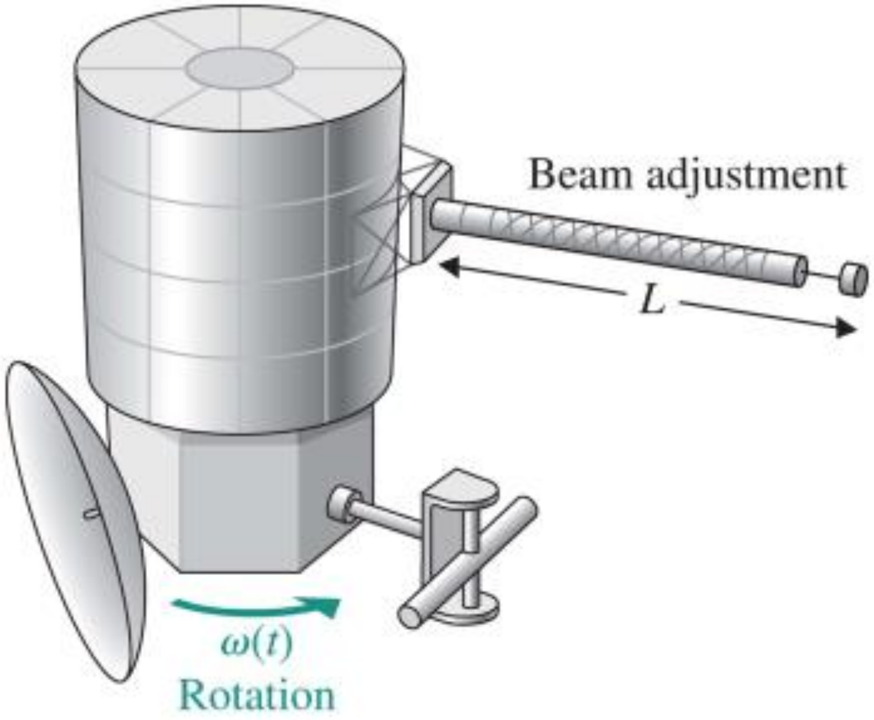
สาขาวิศวกรรมหุ่นยนต์
คณะวิศวกรรมศาสตร์และเทคโนโลยีอุตสาหกรรม
มหาวิทยาลัยราชภัฏสวนสุนันทา

Chapter 2 Mathematic Model of System
Lecture 4 Effect of pole locations

- Understand the concept of transient response
- Observe the influence of the pole location in the system response
- Find the response of a system for a certain input
- Computer Simulation (Matlab/Simulink)

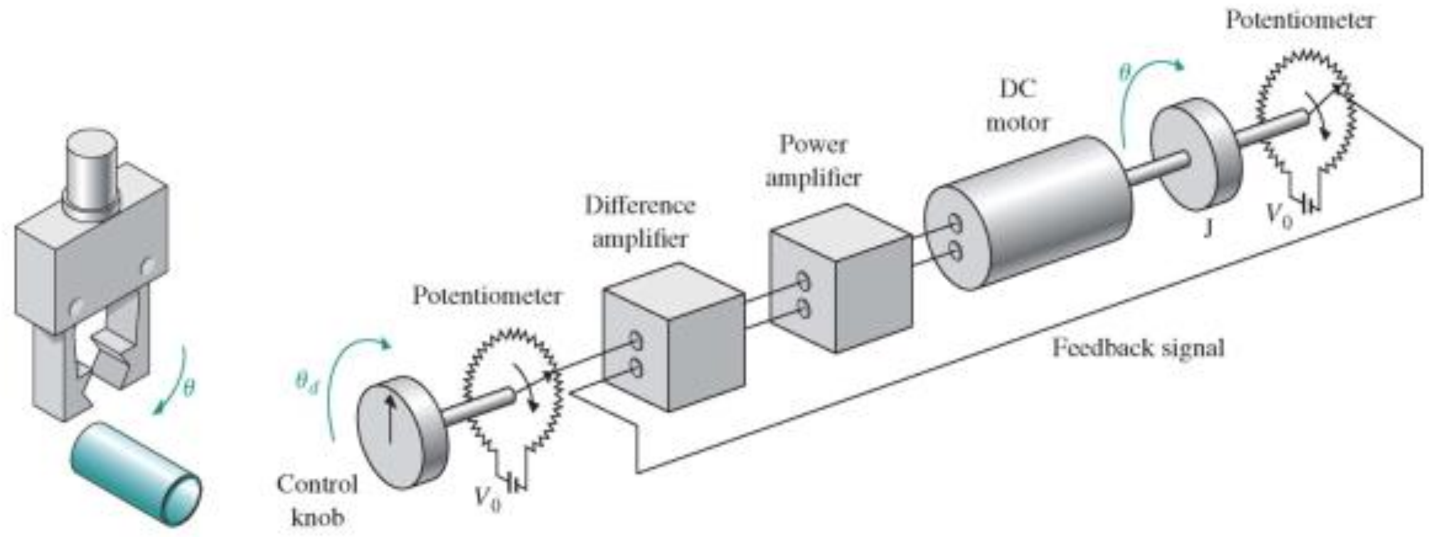
Application

The rotational velocity of of the satellite is adjusted by changing the length of the beam. How can we determine the shape of the transient response?



Applications

A robot gripper is to be controlled by a DC motor. How can we determine the transient response of the gripper's position?



From the last lecture

A transfer function can be written as

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \quad (1)$$

with $n \geq m$. The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\lim_{s \to z_i} N(s) = 0 \quad (2)$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

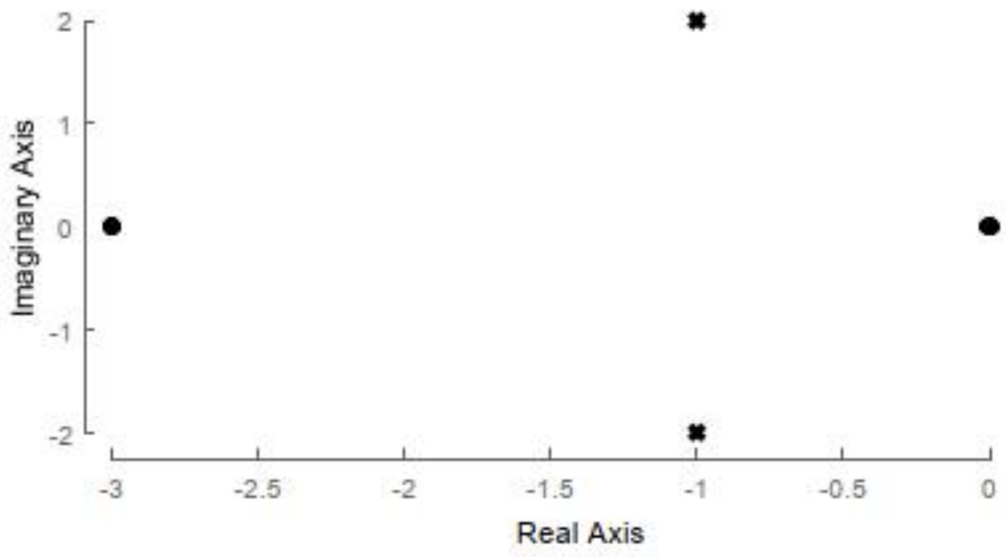
$$\lim_{s \to p_i} H(s) = \infty \quad (3)$$

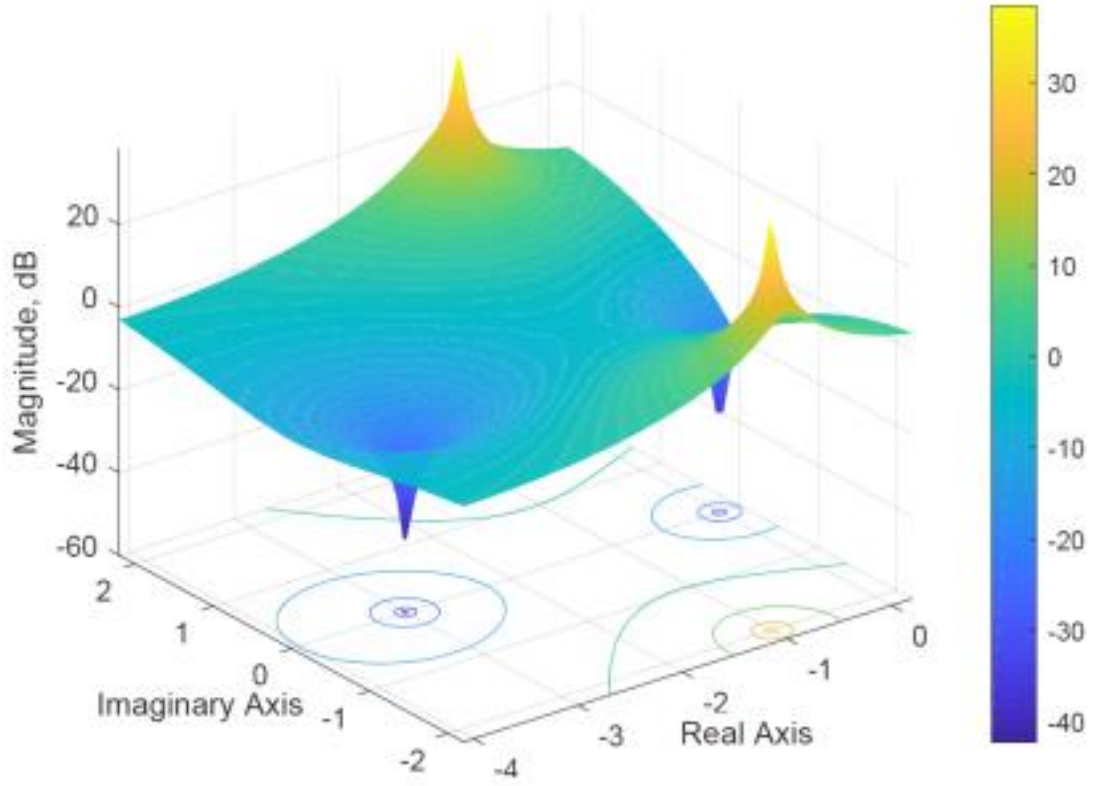
Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2 + 2s + 5}$$

→ Poles: $-1 + 2j$, $-1 - 2j$

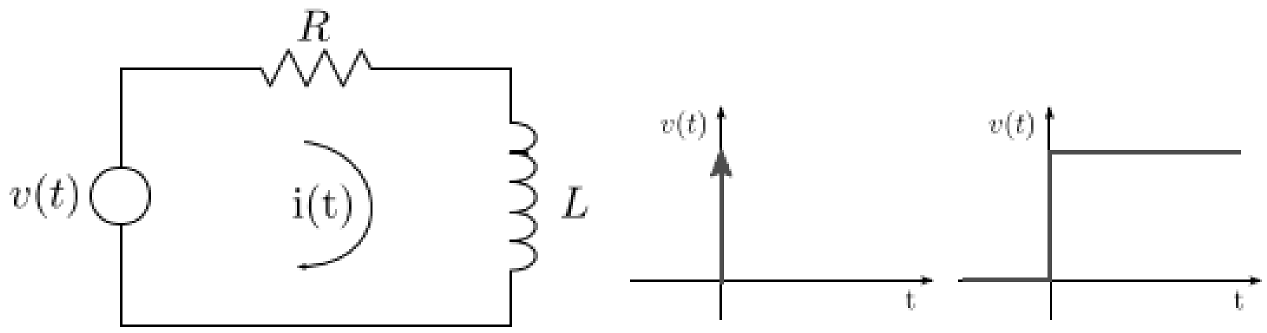
→ Zeros: 0, -3





First order systems

Consider the RL circuit shown.



$$V(s) = (R + Ls)I(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1/R}{s(L/R) + 1}$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \tag{4}$$

→ The denominator must be in the form of $\tau s + \mathbf{1}$

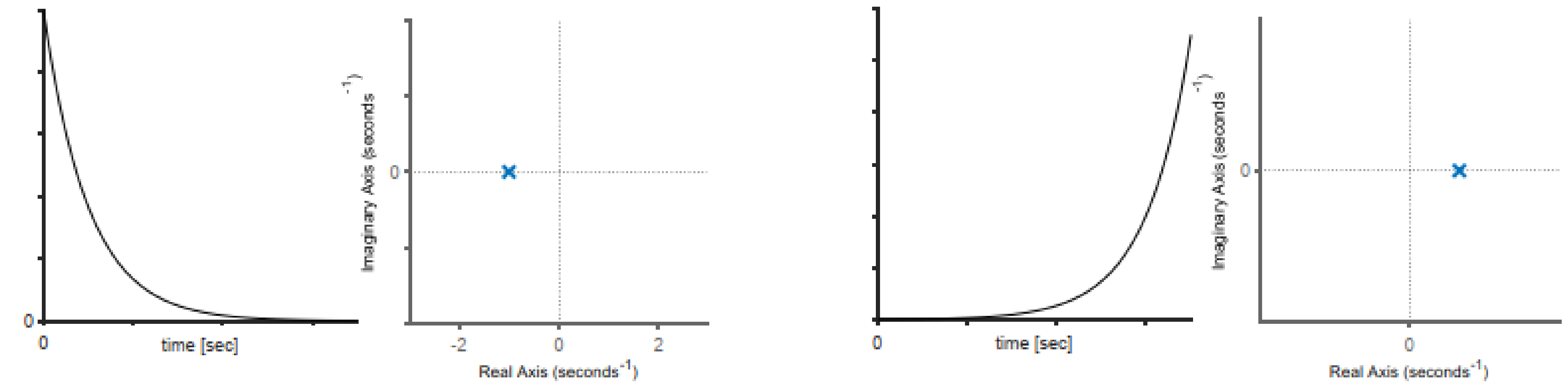
First order transfer functions

Impulse response: $v(t) = \delta(t) \Rightarrow V(s) = 1$

$$I(s) = \frac{1}{R}\frac{1}{\tau s + 1} = \frac{1}{\tau R}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

The pole is $s = -1/\tau$. The time response is:

$$i(t) = \frac{1}{\tau R} e^{-\frac{t}{\tau}}$$



If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

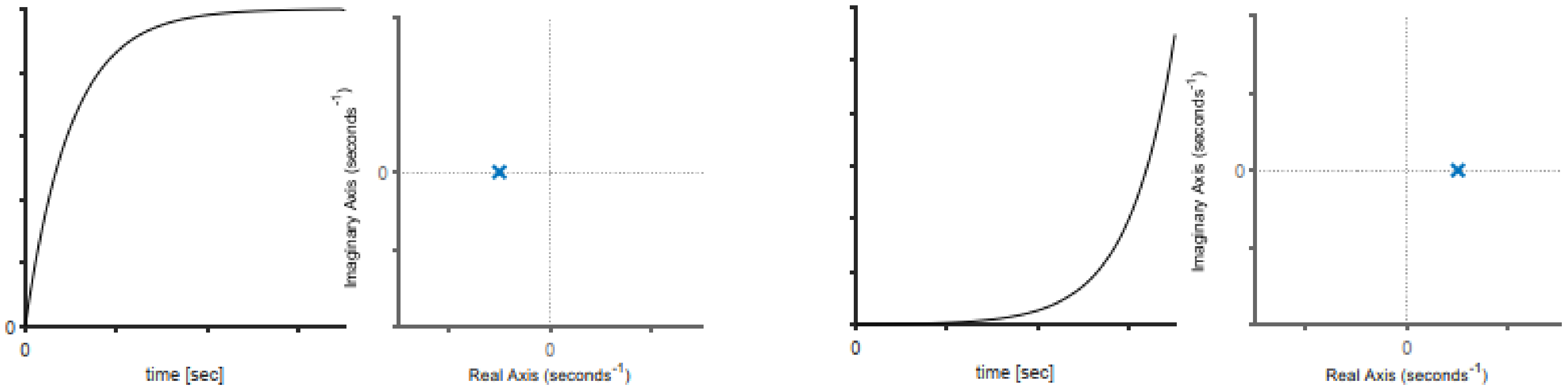
First order transfer functions

Step response: $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$I(s) = \frac{1}{\tau R}\left(\frac{1}{s}\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{k_1}{s} + \frac{k_2}{s+\frac{1}{\tau}}\right)$$

Solving for the partial fraction coefficients: $k_1 = \tau$, $k_2 = -\tau$, thus:

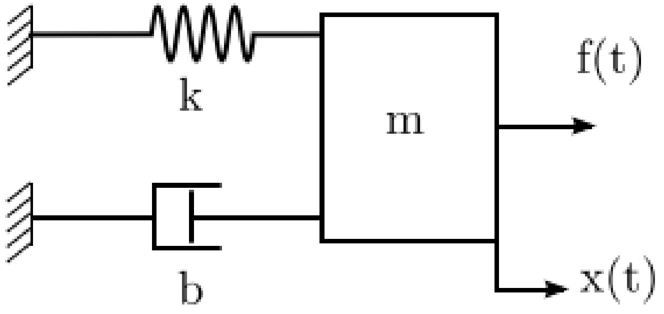
$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

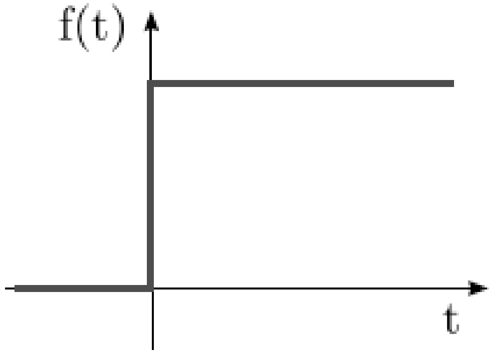


If $\tau > 0$, the pole is on the left-half s-plane.

If $\tau < 0$, the pole is on the right-half s-plane.

Second order transfer functions





Transfer function: Standard form

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{m}\left(\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\right)$$

In standard form we have

$$H(s) = \frac{1}{k}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

where:

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

Second order response

Let us now analyse the response to a step input of a second order system

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{6}$$

The poles of the transfer function are:

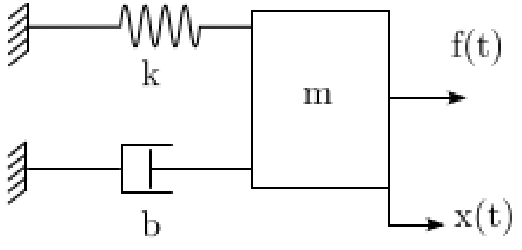
$$s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{7}$$

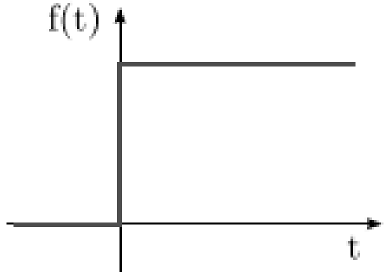
Thus:

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\zeta + j\sqrt{1 - \zeta^2}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\zeta - j\sqrt{1 - \zeta^2}\right)$$

Roots can be real or complex $\Rightarrow$ Inverse transformation ?

Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{8}$$

Case 1: $\zeta \geq 1$, (lots of damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

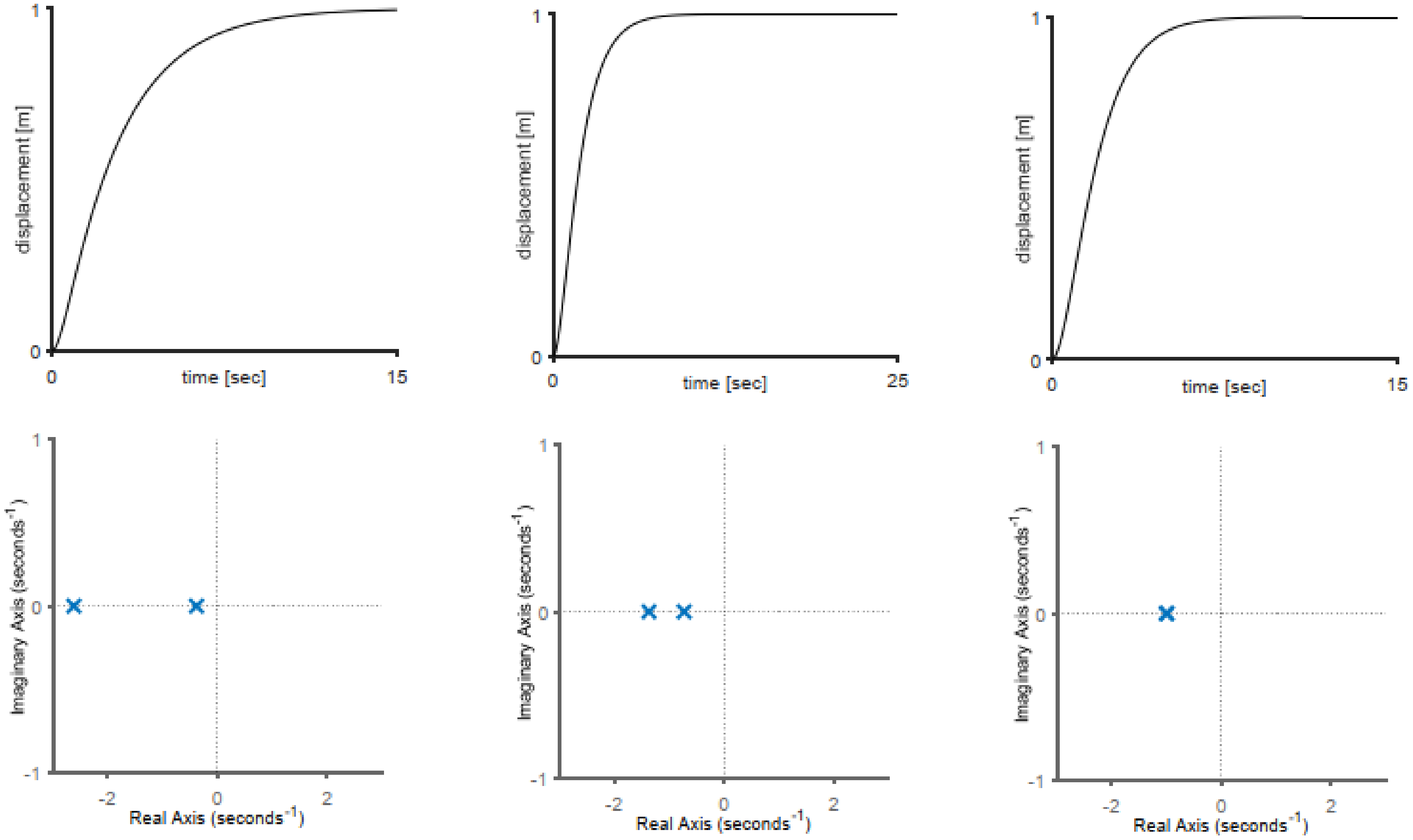
Roots are negative real numbers. Partial fraction expansion yields:

$$X(s) = \frac{1}{m}\left(\frac{k_1}{s} + \frac{k_2}{(s + a_1)} + \frac{k_3}{(s + a_2)}\right) \tag{9}$$

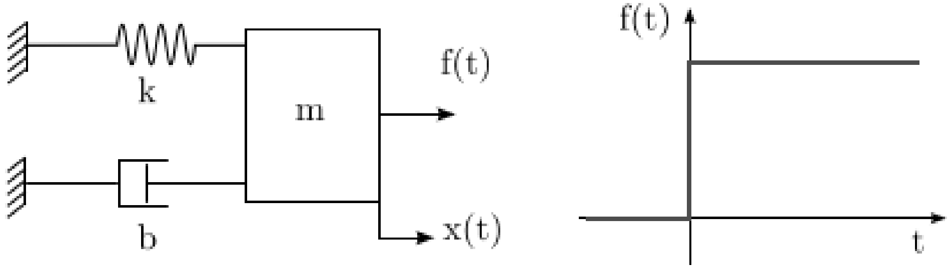
Overdamped system

Example: $m = 1$ kg, $k = 1$ N/m

$b = 3$ Ns/m, $\zeta = 1.5$; $b = 2.1$ Ns/m, $\zeta = 1.05$; $b = 2$ Ns/m, $\zeta = 1$.



Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{10}$$

Case 2: $0 < \zeta < 1$, (some damping)

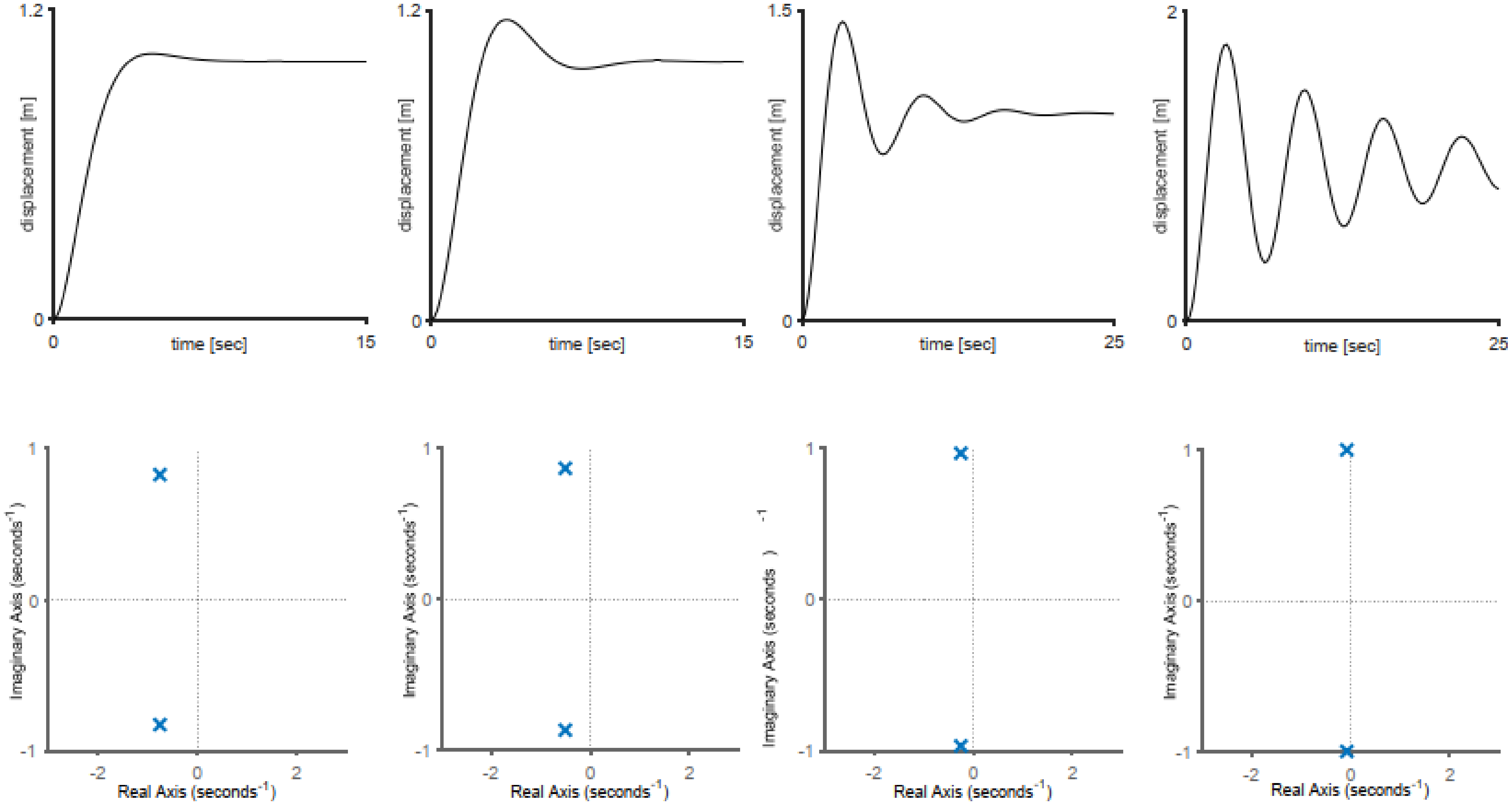
$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

Roots are complex conjugate numbers with a negative real part. Thus:

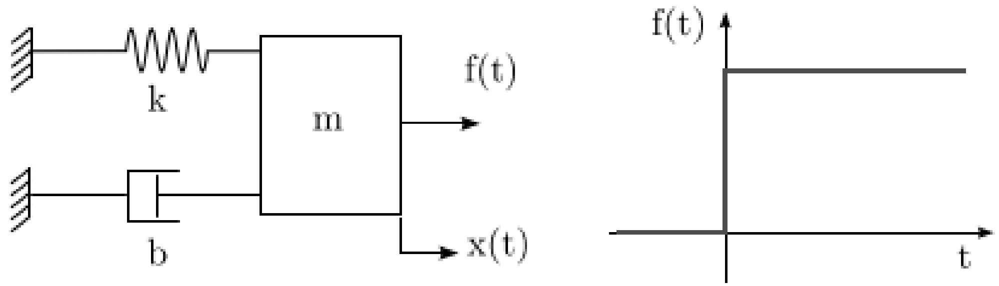
$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \tag{11}$$

Underdamped system

$b = 1.5,\ \zeta = 0.75$ $b = 1,\ \zeta = 0.5;$ $b = 0.5,\ \zeta = 0.25,$ $b = 0.1,\ \zeta = 0.05$



Second order response



$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{12}$$

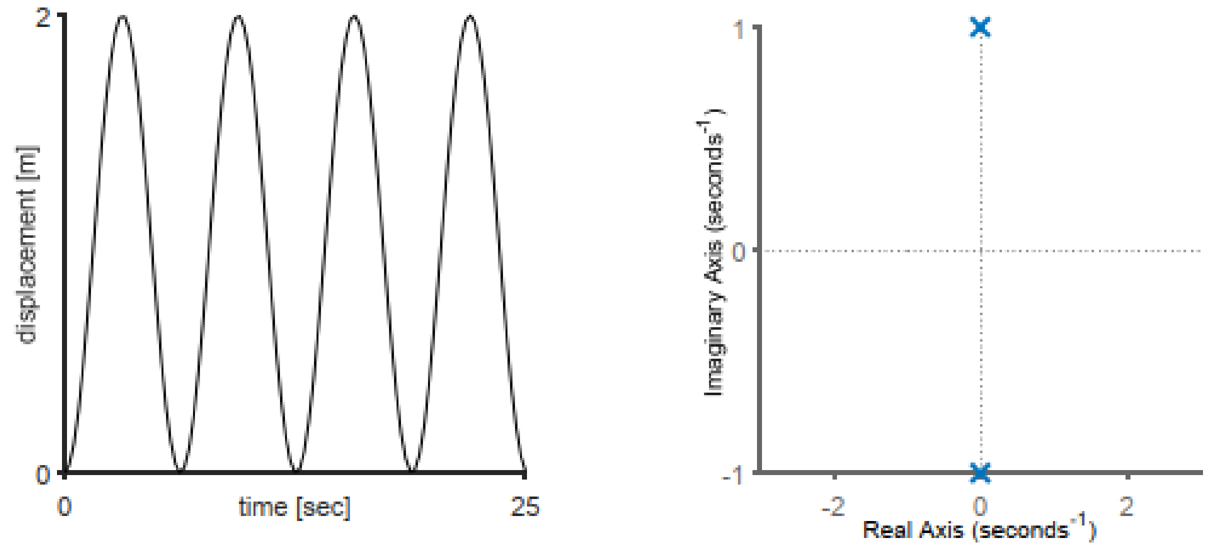
Case 3: $\zeta = 0$, (no damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$$
$$s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

Roots are purely complex conjugate numbers. Thus:

$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) \rightarrow x(t) = \frac{1}{k}[1 - \cos(\omega_n t)] \tag{13}$$

Undamped system



The frequency of oscillation for of an undamped system is called the natural frequency.

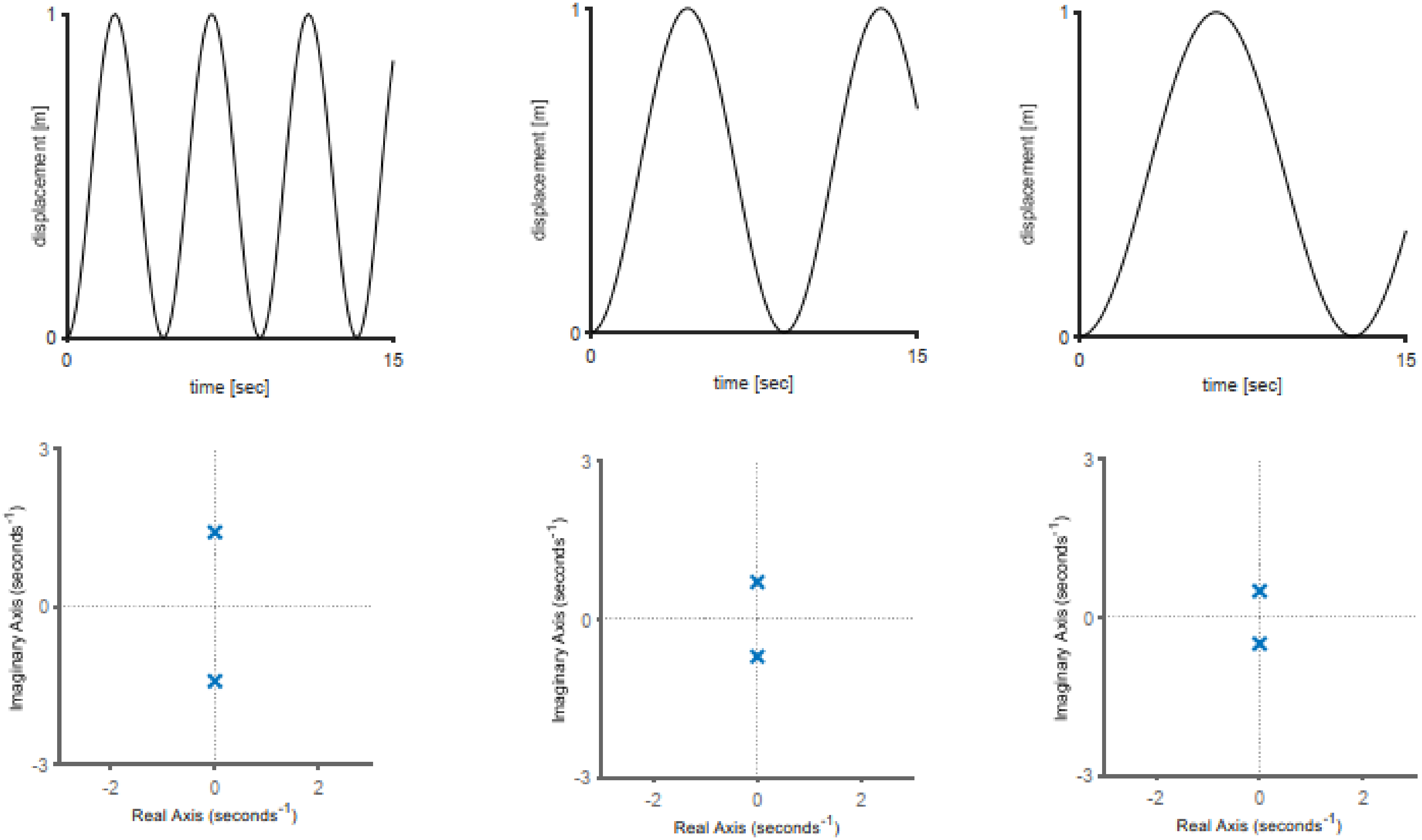
In our example:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{14}$$

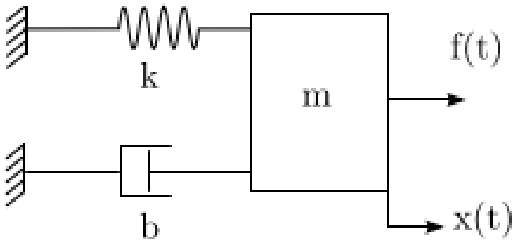
Natural frequency

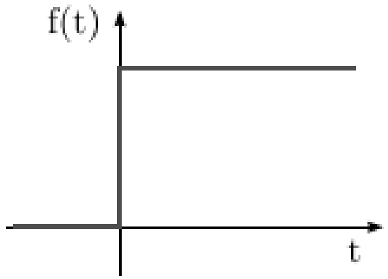
The frequency the system oscillates when $\zeta = 0$. Example: $b = \zeta = 0$, $k = 1$.

$m = 1$, $\omega_n = 1$ rad/s; $m = 2$, $\omega_n = 0.71$ rad/s; $m = 4$, $\omega_n = 0.5$ rad/s;



Second order response





$$X(s) = \frac{1}{k}\left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{15}$$

Case 4: $\zeta < 0$, (hypothetical negative damping)

$$s_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right), \quad s_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

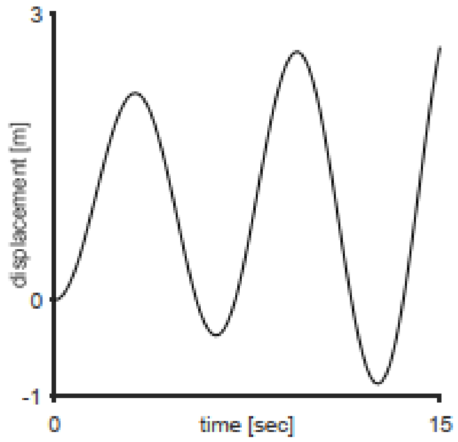
Roots (real or imaginary) have positive real parts. Possible solutions are:

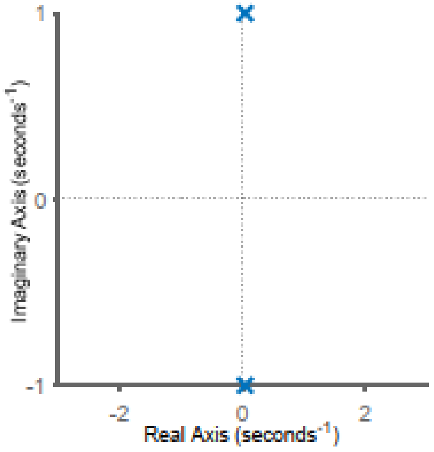
$$x(t) = k(1 + k_2 e^{s_1 t} + k_3 e^{s_2 t}) \tag{16}$$

$$x(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{|\zeta|\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2 t} + \cos^{-1}\zeta\right)\right] \tag{17}$$

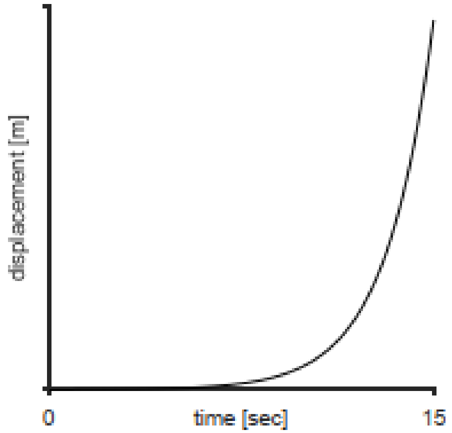
Unstable system

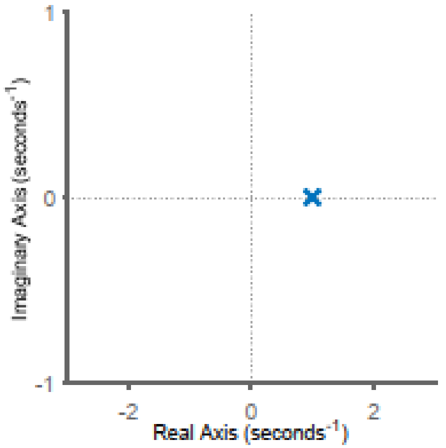
$b = -0.1$ Ns/m;



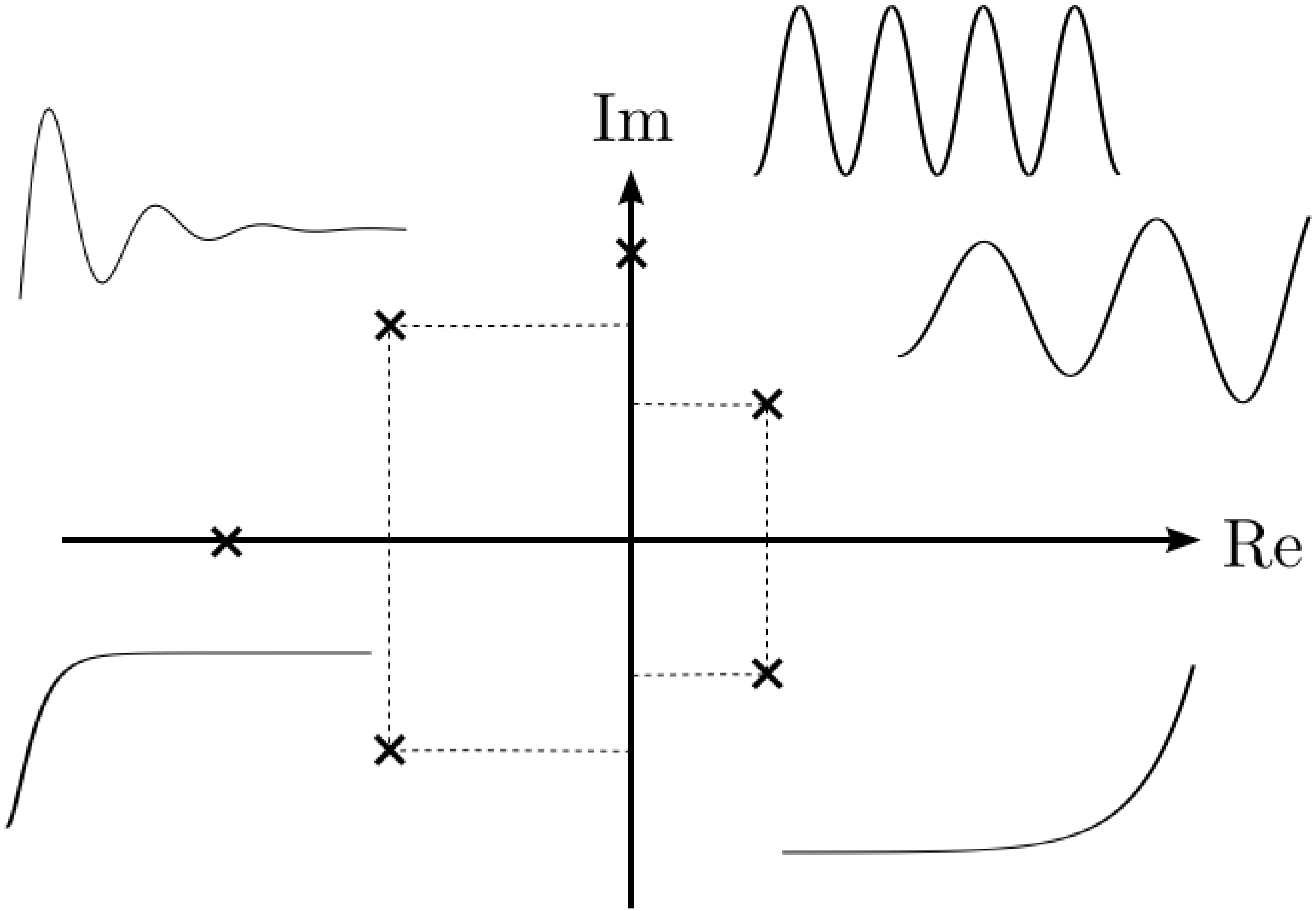


$b < -1$ Ns/m;

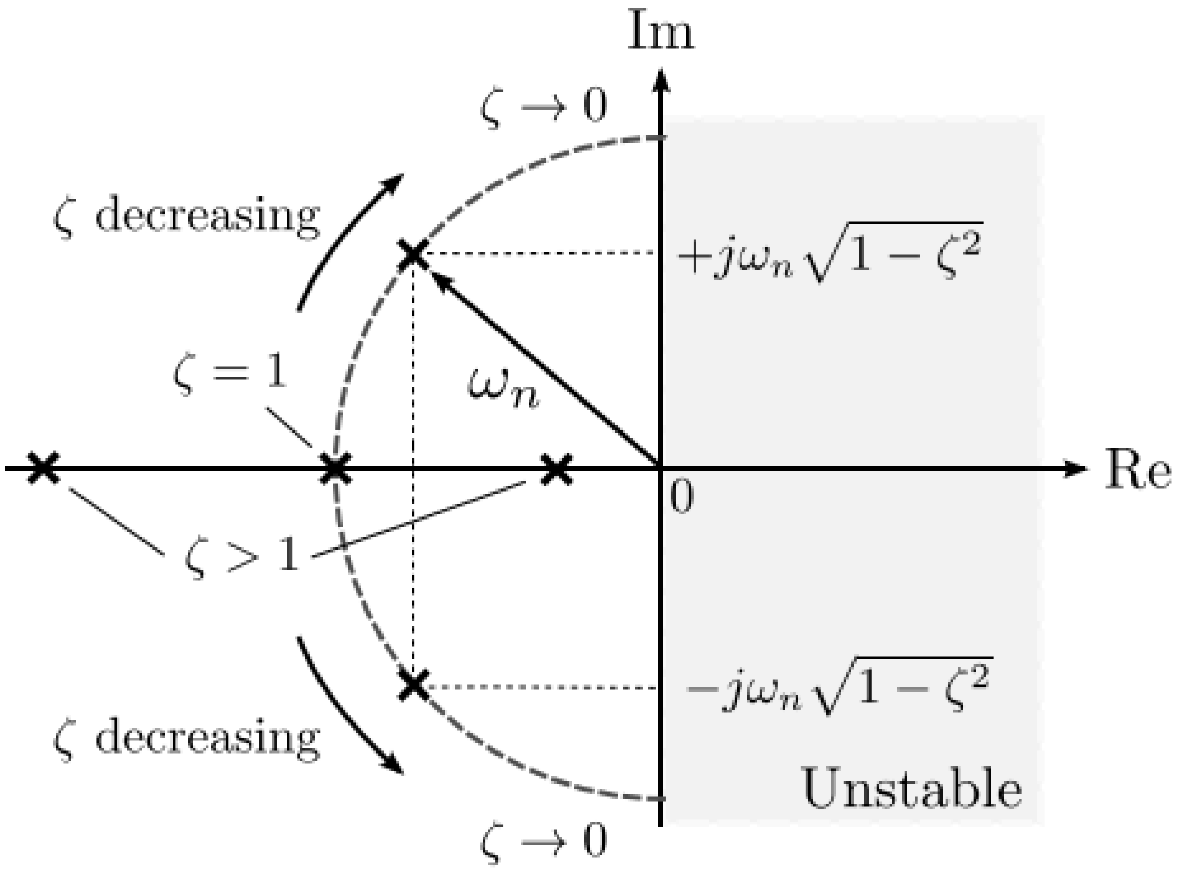




Summary



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



Location of poles in the s-plane

For a second order system, the poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

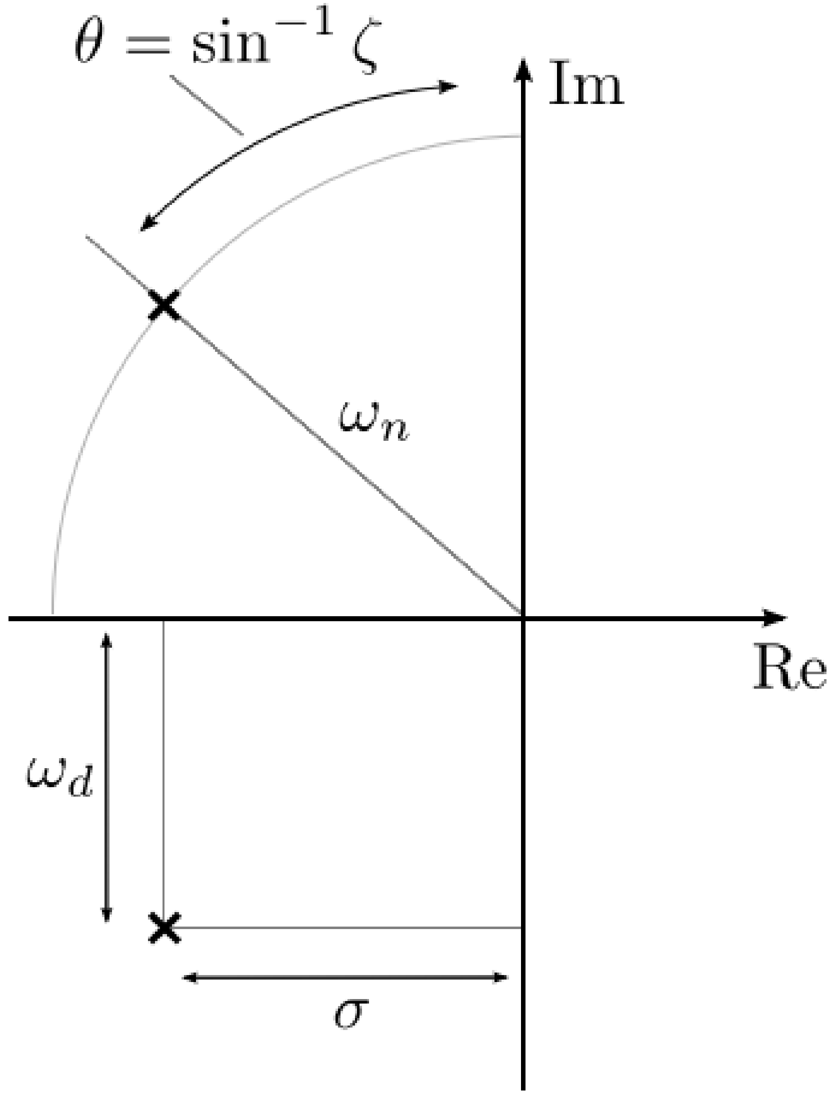
where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

→ The magnitude of s is

$$|s| = \sqrt{(\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2} = \omega_n$$

→ The angle to the imaginary axis is

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} \rightarrow \theta = \sin^{-1}\zeta$$



Exercise 21

Discuss the correlation between the poles of

$$H(s) = \frac{2s+1}{s^2+2s+5} \tag{18}$$

and the impulse response of the system and find the exact impulse response.

Procedure:

→ Calculate the damping ratio and the natural frequency

→ Calculate inverse transform

Exercise 21 - continued

$$H(s) = \frac{2s + 1}{s^2 + 2s + 5}$$

Exercise 21 - continued

Exercise 21 - continued

Exercise 22

Consider the system of the form

$$H(s) = \frac{9}{s^2 + bs + 9} \tag{19}$$

Calculate and sketch the response to an unit step input for the following cases

$\rightarrow b = 9$

$\rightarrow b = 0$

$\rightarrow b = 2$

$\rightarrow b = 6$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 9s + 9} \frac{1}{s} \tag{20}$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 9} \frac{1}{s} \tag{21}$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 2s + 9}\frac{1}{s} \tag{22}$$

Exercise 22 - continued

$$H(s) = \frac{9}{s^2 + 6s + 9} \frac{1}{s} \tag{23}$$

Matlab/Simulink